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A Remark on Reddy's Paper on the Rational Approximation of $(1-x)^{1/2}$

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Recently A. R. Reddy [1] studied the approximation of $(1-x)^{1/2}$ on $[0, 1]$ by rational functions $p(x)/q(x)$ with $p, q \in P(n)$, where $P(n)$ denotes the set of all polynomials in one variable with real nonnegative coefficients and of degrees at most n. Reddy has shown the inequality

$$\|(1-x)^{1/2} - p(x)/q(x)\|_{L_\infty[0,1]} \geqslant (1/4)\, n^{-1/2} \tag{1}$$

for all $p, q \in P(n)$, $q \not\equiv 0$, provided only $n \geqslant 12$. On the other hand for all $n \geqslant 2$ there exist $p_0, q_0 \in P(n)$, $q_0 \not\equiv 0$, with

$$\|(1-x)^{1/2} - p_0(x)/q_0(x)\|_{L_\infty[0,1]} \leqslant 4(\log n)^{1/2}\, n^{-1/2}. \tag{2}$$

The aim of the present note is to improve upon this last assertion by showing that the $(\log n)^{1/2}$-factor in (2) is superfluous, so that the lower estimate (1) is best possible up to the numerical value of the constant factor of $n^{-1/2}$. More precisely the following result will be proved.

THEOREM. *For all nonnegative integers n one has*

$$\left\|(1-x)^{1/2} - \left(\sum_{k=0}^{n} \binom{2k}{k} \left(\frac{x}{4}\right)^k\right)^{-1}\right\|_{L_\infty[0,1]} = 4^n(2n+1)^{-1}\binom{2n}{n}^{-1} \tag{3}$$

and here the right-hand side is strictly less than $(\pi^{1/2}/2)\, n^{-1/2}$ *for* $n \geqslant 1$.

Proof. Define

$$T_n(x) := \sum_{k=0}^{n} \binom{2k}{k} \left(\frac{x}{4}\right)^k \qquad (n = 0, 1, \ldots), \tag{4}$$

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the nth Taylor polynomial of $(1-x)^{-1/2}$, and further

$$D_n(x) := T_n(x)^{-1} - (1-x)^{1/2} \tag{5}$$

for $x \in [0, 1]$. If one has

$$D_n'(x) > 0 \qquad \text{for all } x \in (0, 1) \text{ and } n = 0, 1, \ldots, \tag{6}$$

it is clear that D_n is strictly increasing on $[0, 1]$ such that

$$0 = D_n(0) \leqslant D_n(x) \leqslant D_n(1) = T_n(1)^{-1}$$

for $n = 0, 1, \ldots, x \in [0, 1]$. This implies (3), since using (4) one shows easily by induction on n that

$$T_n(1) = 4^{-n}(2n+1)\binom{2n}{n} \qquad (n = 0, 1, \ldots).$$

In virtue of (5) the inequalities (6) and

$$(1-x)^{-1/2}\, T_n(x)^2 > 2T_n'(x) \qquad \text{for } x \in (0, 1),\ n = 0, 1, \ldots \tag{7}$$

are equivalent. Defining

$$R_n(x) := (1-x)^{-1/2} - T_n(x) \qquad \text{for real } x < 1$$

one has

$$(1-x)^{-1/2}\, T_n(x)^2 = (1-x)^{-3/2} - 2(1-x)^{-1} R_n(x) + (1-x)^{-1/2} R_n(x)^2.$$

Now the Taylor series $\sum_{k=0}^{\infty} a_k x^k$ for this last function has obviously the following properties:

$$a_k > 0 \qquad \text{for all } k \geqslant 0,$$

and

$$a_k = (-1)^k \binom{-3/2}{k} = 4^{-k}(2k+1)\binom{2k}{k} \qquad \text{for } 0 \leqslant k \leqslant n,$$

since R_n has a zero of order $n+1$ at $x = 0$. Therefore one obtains

$$(1-x)^{-1/2}\, T_n(x)^2 > \sum_{k=0}^{n-1} (2k+1)\binom{2k}{k}\left(\frac{x}{4}\right)^k$$
$$\text{for } x \in (0, 1),\ n = 0, 1, \ldots$$

and here the right-hand side (being 0 for $n = 0$) is exactly $2T_n'(x)$, which gives (7) and thus completes the proof of the first assertion of the theorem.

Finally the sequence $\{4^n n^{1/2}(2n+1)^{-1}\binom{2n}{n}^{-1}\}_{n=0,1,\ldots}$ is strictly increasing and converges to $\pi^{1/2}/2$ by Stirling's formula.

It should be remarked that by (1) and (3) each element of the sequence

$$\{n^{1/2} \inf_{\substack{(p,q)\\ p,q\in P(n),q\neq 0}} \|(1-x)^{1/2}-p(x)/q(x)\|_{L_\infty[0,1]}\}_{n=1,2,\ldots}$$

lies in the interval $[1/4, \pi^{1/2}/2)$. Therefore it would be interesting to study this sequence in more detail, e.g., to determine its lim inf and lim sup.

In the first version of this note it was shown by another method of proof that the norm in (3) is less than $3n^{-1/2}$ for $n=1,2,\ldots$. The author would like to thank G. Meinardus for the communication of his conjecture that (3) could be true.

REFERENCE

1. A. R. REDDY, A note on rational approximation to $(1-x)^{1/2}$, *J. Approx. Theory* **25** (1979), 31–33.